

Axisymmetric multiwormholes revisited

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Abstract

The construction of stationary axisymmetric multiwormhole solutions to gravitating field theories admitting toroidal reductions to three-dimensional gravitating sigma models is reviewed. We show that, as in the multi-black hole case, strut singularities always appear in this construction, except for very special configurations with an odd number of centers. We also review the analytical continuation of the multicenter solution across the n cuts associated with the wormhole mouths. The resulting Riemann manifold has 2^n sheets interconnected by $2^{n-1}n$ wormholes. We find that the maximally extended multicenter solution can never be asymptotically locally flat in all the Riemann sheets.

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1 Introduction

Black holes attract each other, so stationary multi-black hole configurations should not be possible in principle. Notwithstanding this, stationary axisymmetric multi-black-hole solutions to Einstein's equations were built long ago [1] by linearly superposing aligned Schwarzschild solutions. Such solutions are possible only because of the presence of strut-like conical singularities along the axis, which account for the extra balancing forces [2, 3]. However balance without struts is possible if both positive and negative charges are allowed, as discussed in [2, 4] where regular multi-center solutions with an odd number of carefully chosen and positioned sources were constructed.

Besides black hole solutions and solutions with naked singularities, some gravitating field theories also admit spherically symmetric Lorentzian wormhole solutions, the first known being due to Ellis [5] and Bronnikov [6]. As in the black hole case, it should be possible to linearly superpose two or more aligned wormhole solutions to obtain a stationary axisymmetric multi-wormhole configuration. Indeed, this technique was applied some time ago to the generation of axisymmetric multi-wormhole solutions to five-dimensional general relativity [7] from the Chodos-Detweiler (CD) spherically symmetric wormhole solution [8]. After reduction to four dimensions, the CD wormhole is massless and electrically charged, so two such wormholes should repel each other. Nevertheless, it was argued in [7] that the strut singularities should be altogether absent in the multi-wormhole case. A similar construction – and a similar argument – was applied in [9] to the Bronnikov wormhole solution to four-dimensional Einstein-Maxwell theory with a massless phantom scalar field (coupled repulsively to gravitation), and generalized in [10] to wormhole solutions with NUT charge to the same theory.

Recently, the supercritically charged Reissner-Nordström-NUT (RN-NUT) solution to the Einstein-Maxwell equations was shown to correspond to a geodesically complete, traversable Lorentzian wormhole [11]. Due to the presence of a NUT charge, the RN-NUT metric is not asymptotically Minkowskian. A possible cure for this would be to consider it as a building block from which to construct a NUT-anti-NUT, two-wormhole solution. The two monopole NUT charges would compensate, to give a net dipole charge (angular momentum) at infinity. A solution built up from two spherically symmetric wormholes should be axisymmetric, so it should be possible to construct it using the technique of [1, 7], if all the charges associated with the second wormhole were opposite to those of the first wormhole. The sum of the attractive forces between opposite electric and magnetic charges and of the repulsive forces between opposite masses and NUT charges would then

lead, because the RN-NUT wormholes are supercritically charged, to a net attractive force, so that struts should be necessary for balance.

It therefore seems necessary to reconsider the construction of stationary axisymmetric multiwormhole solutions and the issue of strut singularities. In the next section we review the construction of wormhole solutions to gravitating field theories as geodesic solutions to their three-dimensional toroidal reductions. We then revisit in Sect. 3 the construction of aligned multiwormhole solutions from linear superpositions of harmonic functions centered on the symmetry axis. We show why the naive regularity argument of [7] fails, so that struts holding the wormholes in place necessarily appear in this construction, except for certain fine-tuned configurations with an odd number of centers. Analytical continuation to a multi-sheeted Riemann manifold is discussed in Sect. 4. We find that struts can never be avoided in the maximally extended spacetime, which is always asymptotically conical in some of the Riemann sheets. Our results are summarized in the concluding section.

2 Wormholes from target space geodesics

Consider a four-dimensional gravitating field theory, possibly obtained by toroidal dimensional reduction from some higher-dimensional theory, involving a set of gravity-coupled scalar fields ϕ^a and $U(1)$ vector fields A^I . The stationary sector of the theory may be further reduced to three dimensions by assuming the generic stationary metric parametrization

$$ds^2 = -f(dt + \mathcal{A}_i dx^i)^2 + f^{-1}h_{ij} dx^i dx^j, \quad (2.1)$$

where the various fields depend only on the three reduced space coordinates x^i ($i = 1, 2, 3$). Solving the mixed $G_0^i = T_0^i$ Einstein equations enables to trade the Kaluza-Klein vector field \mathcal{A}_i for a three-dimensional scalar twist potential χ , while the four-dimensional $U(1)$ vector fields may similarly be traded for pairs of scalar electric v^I and magnetic u^I potentials. This reduction leads to the gravitating three-dimensional σ -model action [12]

$$S_{(3)} = \int d^3x \sqrt{|h|} \left[-R_{(3)} + G_{AB}(X) \partial_i X^A \partial_j X^B h^{ij} \right], \quad (2.2)$$

where $R_{(3)}$ is the Ricci scalar built out of the h_{ij} . The equations of motion for the scalars $X^A = (f, \chi, v^I, u^I, \phi^a)$

$$\frac{1}{\sqrt{h}} \partial_i \left(\sqrt{h} h^{ij} G_{AB} \partial_j X^B \right) = 0 \quad (2.3)$$

define a harmonic map from the three-space $\{x^i\}$ to the target space or potential space $\{X^A\}$ with the line element

$$dS^2 = G_{AB} dX^A dX^B. \quad (2.4)$$

In many cases of interest, this target space is a symmetric space or coset (for instance $SU(2,1)/S[U(1,1) \times U(1)]$ in the Einstein-Maxwell case, or $SL(3,R)/SO(2,1)$ in the case of five-dimensional general relativity), which leads to fruitful solution-generating techniques. The symmetric space property will not be used in the present paper.

If now we assume that the coordinates X^A depend on a single potential function $\sigma(\vec{x})$, we have the freedom to choose this potential to be harmonic [13],

$$\partial_i(\sqrt{h}h^{ij}\partial_j\sigma) = 0. \quad (2.5)$$

The sigma-model field equations (2.3) then reduce to the geodesic equations for the target space (2.4), which are first integrated by

$$G_{AB} \frac{dX^A}{d\sigma} \frac{dX^B}{d\sigma} = 2\epsilon \quad (2.6)$$

(ϵ constant), while the three-dimensional Einstein equations reduce to

$$R_{(3)ij} = 2\epsilon \partial_i\sigma\partial_j\sigma. \quad (2.7)$$

The target space metric is indefinite, the two gravitational potentials f and χ and the original scalar fields ϕ^a contributing positively to the metric (2.4) and the electric and magnetic potentials v^I and u^I contributing negatively (the signs are reversed in the case of phantom fields). Accordingly, the geodesics fall in three classes depending on the value of ϵ : a timelike class ($\epsilon > 0$), which includes black hole solutions; a null class ($\epsilon = 0$), which corresponds to extremal black holes; and a spacelike class ($\epsilon < 0$), which includes wormhole solutions. Choosing in this last case the convenient normalization $\epsilon = -1$, and assuming spherical symmetry, the coupled equations (2.5) and (2.7) are solved by the harmonic function and reduced space metric

$$\sigma = \arctan\left(\frac{b}{r}\right), \quad (2.8)$$

$$dl^2 \equiv h_{ij}dx^i dx^j = dr^2 + (r^2 + b^2)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (2.9)$$

Note that the reduced three-space has the wormhole topology, the coordinate r varying in the whole real axis, with two asymptotic regions $r = \pm\infty$,

and a wormhole neck of radius b at $r = 0$. This necessary condition for the four- or higher-dimensional metric to correspond to a Lorentzian wormhole is not sufficient, in addition the gravitational potentials f and χ (as well as the Kaluza-Klein potentials and moduli in the higher-dimensional case) must also be regular for all r . The case of the CD wormhole solution to five-dimensional general relativity is treated in [7]. In the case of the RN-NUT wormhole solution to the four-dimensional Einstein-Maxwell equations, the solution of the target space geodesic equations yields for the complex Ernst potentials $\mathcal{E} = f + i\chi - \bar{\psi}\psi$ and $\psi = v + iu$:

$$\mathcal{E} = \frac{1 - \mu \tan \sigma}{1 + \mu \tan \sigma}, \quad \psi = \frac{\gamma \tan \sigma}{1 + \mu \tan \sigma}, \quad (2.10)$$

where the complex parameters μ and γ of the geodesic under consideration are related by

$$|\gamma^2| - |\mu^2| = 1. \quad (2.11)$$

For the harmonic function (2.8) this gives

$$f(r) = \frac{r^2 + b^2}{(r + m)^2 + n^2}, \quad \chi(r) = \frac{-2nr}{(r + m)^2 + n^2}, \quad (2.12)$$

with

$$b\mu = m + in, \quad b\gamma = q - ip \quad (b^2 = q^2 + p^2 - m^2 - n^2). \quad (2.13)$$

The four-dimensional metric (2.1) with NUT parameter n then corresponds to a wormhole with a neck of areal radius n at $r = -m$.

3 Multicenter configurations

More generally, assuming only axisymmetry of the three-dimensional metric, which may be written in the Weyl form as

$$dl^2 = e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2, \quad (3.1)$$

the harmonicity condition (2.5) on $\sigma(\rho, z)$ and equation (2.7) read

$$\rho^{-1}(\rho\sigma_{,\rho}), \rho + \sigma_{,z,z} = 0, \quad (3.2)$$

$$\partial_\zeta k = -2\rho(\partial_\zeta \sigma)^2, \quad (3.3)$$

where we have combined the two cylindrical coordinates ρ and z into the complex variable

$$\zeta = \rho + iz. \quad (3.4)$$

Because of the linearity of (3.2), any number of spherically symmetric solutions σ_p of this equation may be superposed to yield an axisymmetric solution $(\mathcal{E}, \psi, h_{ij})$ depending on $\sigma = \sum_p \sigma_p$. In the case of the RN-NUT wormhole, the resulting four-dimensional metric will be (2.1), with the reduced metric given by (3.1) and (3.3), and

$$f = \frac{1 + \tan^2 \sigma}{|1 + \mu \tan \sigma|^2}, \quad \partial_\zeta \mathcal{A}_\varphi = 2i \operatorname{Im} \mu \rho \partial_\zeta \sigma, \quad (3.5)$$

so that the Kaluza-Klein potential \mathcal{A}_φ will also be a linear superposition.

The first step is to cast the spherically symmetric reduced metric (2.9) in the Weyl form (3.1). Following [7] we first introduce a new radial coordinate $R > 0$ related to r by

$$r = R - \frac{b^2}{4R}, \quad (3.6)$$

leading to the isotropic form of the solution (2.8)-(2.9)

$$\sigma = 2 \arctan \left(\frac{b}{2R} \right), \quad (3.7)$$

$$dl^2 = \left(1 + \frac{b^2}{4R^2} \right)^2 \left[dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (3.8)$$

The metric (3.8) may then be transformed to the Weyl form by the conformal Joukovski transformation

$$\zeta = \chi + \frac{b^2}{4\chi}, \quad (3.9)$$

where we have put

$$\chi = iRe^{-i\theta}. \quad (3.10)$$

The harmonic function σ and the Weyl metric function k are then given by the complex analytical continuation $m \rightarrow ib$ of the formulas valid for the Schwarzschild solution [14]

$$\sigma = \arctan \left(\frac{b}{\operatorname{Re} \eta} \right), \quad e^{2k} = \frac{(\operatorname{Re} \eta)^2 + b^2}{|\eta|^2}, \quad (3.11)$$

with

$$\eta^2 = (\zeta - b)(\bar{\zeta} + b). \quad (3.12)$$

The square root of (3.12) can be written explicitly as

$$\eta = |\chi|^{-1} \left(\chi - \frac{b}{2} \right) \left(\bar{\chi} + \frac{b}{2} \right), \quad (3.13)$$

where $\chi(\zeta)$ is given by the inverse function of (3.9),

$$\chi = \frac{1}{2}[\zeta + (\zeta^2 - b^2)^{1/2}]. \quad (3.14)$$

This inverse function is bivalued, with two determinations χ^+ and χ^- corresponding to the two determinations of the square root function, and defines a two-sheeted Riemann surface, the two sheets (corresponding to the two sides of the wormhole) being connected along the cut C ($z = 0$, $-b \leq \rho \leq b$) (the wormhole neck).

The superposition of two harmonic functions $\sigma_1(\rho, z - a_1; b_1)$ and $\sigma_2(\rho, z - a_2; b_2)$ (we assume in the following $a_1 < a_2$) yields the harmonic function

$$\sigma = \sigma_1 + \sigma_2 = \arctan \left(\frac{b_2 \operatorname{Re} \eta_1 + b_1 \operatorname{Re} \eta_2}{\operatorname{Re} \eta_1 \operatorname{Re} \eta_2 - b_1 b_2} \right), \quad (3.15)$$

with

$$\eta_p^2 = (\zeta_p - b_p)(\bar{\zeta}_p + b_p) \quad (\zeta_p = \rho + i(z - a_p)), \quad (3.16)$$

and the Weyl metric function

$$k = k_1 + k_2 + 2k_{12}, \quad (3.17)$$

where k_p are the Weyl functions (3.11) for the respective harmonic functions σ_p , and k_{12} is given by the analytical continuation of the formula given in [1, 2, 3, 4]

$$e^{2k_{12}} = \left| \frac{2\eta_1 \bar{\eta}_2 + (\zeta_1 - b_1)(\bar{\zeta}_2 - b_2) + (\zeta_2 + b_2)(\bar{\zeta}_1 + b_1)}{2\eta_1 \eta_2 + (\zeta_1 - b_1)(\bar{\zeta}_2 + b_2) + (\zeta_2 - b_2)(\bar{\zeta}_1 + b_1)} \right|, \quad (3.18)$$

In the special case of the superposition of a wormhole centered at ($\rho = 0$, $z = -a$) and an antiwormhole centered at ($\rho = 0$, $z = a$), $b_1 = -b_2 = b$ and $a_1 = -a_2 = -a$.

The integration constant in (3.18) has been chosen so that the boundary condition

$$k(\infty) = 0, \quad (3.19)$$

which ensures that the reduced three-space is asymptotically Euclidean, is satisfied. It was argued in [7] that, because from (3.3) $k(\rho, z)$ is constant on the symmetry axis $\rho = 0$, this also ensures that the regularity condition (absence of conical singularities)

$$k(0, z) = 0 \quad (3.20)$$

is satisfied for all z . However this argument overlooked the fact that the functions $\chi_p(\zeta)$, and therefore also the functions $\eta_p(\zeta)$ given by (3.16), are discontinuous along the cuts C_p . Indeed these discontinuities are similar to the disk discontinuities observed in the superposition of a ring and a homogeneous field [15] or a Chazy-Curzon particle [3]. In the present multiwormhole case, the metric can be smoothly continued through the disks into other Riemann sheets (see next section), so that these discontinuities are not associated with matter sources. However they do lead to step-function jumps in $k_{12}(0, z)$ when a cut is crossed. It is clear from (3.16) that, on the symmetry axis, the values of the two determinations of the function $\eta_p(\rho, z)$ just above and just below the cut C_p are related by $\eta_p^+(0, a_p) = -\eta_p^-(0, a_p)$. While $k_{12}(0, z) = 0$ for $z < a_1$ or $z > a_2$ (below the two cuts or above the two cuts), a careful computation to order ρ^2 leads to the value (the analytical continuation of the result of [2, 3, 4])

$$e^k(0, z) = e^{2k_{12}}(0, z) = \frac{(b_1 + b_2)^2 + (a_1 - a_2)^2}{(b_1 - b_2)^2 + (a_1 - a_2)^2}, \quad (3.21)$$

on the segment $a_1 < z < a_2$ between the two cuts. This means that this segment corresponds to a strut with positive (for two charges b_1 and b_2 of the same sign) or negative (for two charges of opposite signs) tension keeping the two wormhole mouths apart. In the case of the RN-NUT wormhole with $b_1 + b_2 = 0$, this strut coincides with the Dirac-Misner string connecting the two oppositely charged wormhole mouths.

The above construction is easily generalized to the superposition of n harmonic functions,

$$\sigma = \sum_{p=1}^n \sigma_p, \quad (3.22)$$

leading to the Weyl function

$$k = \sum_{p=1}^n k_p + 2 \sum_{q=2}^n \sum_{p=1}^{q-1} k_{pq}. \quad (3.23)$$

As in the multi-black-hole case, a strutless, regular multiwormhole configuration can be achieved in the case of an odd number of suitably charged and positioned wormholes. We consider only the example of three equally spaced wormholes with $(a_1, a_2, a_3) = (-a, 0, a)$, and $(b_1, b_2, b_3) = (b, c, b)$. Between the cuts C_1 and C_2 , $k_{23}(0, z) = 0$, so that

$$e^k(0, z) = e^{2k_{12}}(0, z) e^{2k_{13}}(0, z). \quad (3.24)$$

By reason of symmetry, the result is the same between \mathcal{C}_2 and \mathcal{C}_3 , so that the solution is regular if

$$\frac{(b+c)^2 + a^2}{(b-c)^2 + a^2} \frac{b^2 + a^2}{a^2} = 1, \quad (3.25)$$

which is solved by

$$a^2 = -\frac{b(b+c)^2}{b+4c}, \quad (3.26)$$

provided $b(b+4c) < 0$. For instance, a configuration with net monopole charges equal to zero, $2b+c=0$, will be in equilibrium for an intercut distance $a = b/\sqrt{7}$.

4 Analytical continuation

Now we discuss the analytical continuation of the multiwormhole solution across the n cuts C_p . To this end we denote the two possible determinations of the inverse Joukowski function by

$$\chi_p^\pm = \frac{1}{2}[\zeta_p \pm (\zeta_p^2 - b_p^2)^{1/2}]. \quad (4.1)$$

Analytical continuation across the cut C_p corresponds to the replacement

$$\chi_p^\pm \longrightarrow \chi_p^\mp = \frac{b_p^2}{4\chi_p^\pm}. \quad (4.2)$$

Accordingly, the complex function η_p^\pm and the real harmonic function σ_p^\pm (defined modulo π) are replaced by

$$\eta_p^\mp = -\eta_p^\pm, \quad \sigma_p^\mp = -\sigma_p^\pm. \quad (4.3)$$

The maximally extended three-dimensional manifold has 2^n Riemann sheets interconnected by $2^{n-1}n$ wormholes [7].

Consider first the case $n=2$, and label the four sheets $(++)$, $(+-)$, $(-+)$, and $(--)$. The respective functions $\sigma(\zeta)$ and $k_{12}(\zeta)$ are

$$\begin{aligned} \sigma^{++} &= -\sigma^{--} = \sigma_1 + \sigma_2, & k_{12}^{++} &= k_{12}^{--} = k_{12}, \\ \sigma^{+-} &= -\sigma^{-+} = \sigma_1 - \sigma_2, & k_{12}^{+-} &= k_{12}^{-+}, \end{aligned} \quad (4.4)$$

with

$$e^{2k_{12}^{+-}} = \left| \frac{-2\eta_1\bar{\eta}_2 + (\zeta_1 - b_1)(\bar{\zeta}_2 - b_2) + (\zeta_2 + b_2)(\bar{\zeta}_1 + b_1)}{-2\eta_1\eta_2 + (\zeta_1 - b_1)(\bar{\zeta}_2 + b_2) + (\zeta_2 - b_2)(\bar{\zeta}_1 + b_1)} \right| \quad (4.5)$$

(the functions k_p , which are even in η_p , are unchanged). In the sheet $(+-)$ or $(-+)$, the effective charges are now $(b_1, -b_2)$ or $(-b_1, b_2)$, so that inter-wormhole forces which are repulsive (attractive) in the first sheet $(++)$ become attractive (repulsive) in those sheets.

By continuation through one or the other cut, we expect that the Weyl function $k^{+-}(\rho, z)$ will now satisfy the regularity condition (3.20) on the segment of the symmetry axis $a_1 < z < a_2$ between the two cuts, and take the value (3.21) on the complementary segments $z < a_1$ and $z > a_2$, which now act as struts counterbalancing the net attractive (repulsive) force between the effective charges. It follows that $k(0, \pm\infty)$ no longer vanishes, so that the boundary condition (3.19) cannot be satisfied. Indeed, direct computation of (4.5) in the limit $(\rho \rightarrow \infty, z \rightarrow \infty)$ yields the value

$$k^{+-}(\infty) = 2k_{12}^{+-}(\infty) = \ln \frac{(b_1 + b_2)^2 + (a_1 - a_2)^2}{(b_1 - b_2)^2 + (a_1 - a_2)^2}, \quad (4.6)$$

in accordance with the value (3.21) on the struts $z < a_1$ and $z > a_2$. Thus the reduced three-space cannot be asymptotically Euclidean in all the Riemann sheets. If, as we have assumed, the Euclidean boundary condition (3.19) holds in the sheets $(++)$ and $(--)$, then the reduced three-metric is asymptotically conical in the sheets $(+-)$ and $(-+)$.

In spite of this discouraging result, one could hope that the strutless three-center configurations discussed in the preceding section would survive analytical continuation, or at the very least that some three-center configuration could be found to be asymptotically locally flat in all the Riemann sheets. However the answer turns out to be negative. Consider a generic three-center configuration obeying the Euclidean boundary condition (3.19) in the sheets $(+++)$ and $(---)$. Then in the other sheets we find

$$\begin{aligned} k^{-++}(\infty) &= k^{+--}(\infty) = 2[k_{12}^{+-}(\infty) + k_{13}^{+-}(\infty)], \\ k^{+-+}(\infty) &= k^{-+-}(\infty) = 2[k_{23}^{+-}(\infty) + k_{12}^{+-}(\infty)], \\ k^{+--}(\infty) &= k^{-++}(\infty) = 2[k_{13}^{+-}(\infty) + k_{23}^{+-}(\infty)]. \end{aligned} \quad (4.7)$$

These can simultaneously vanish only if $k_{12}^{+-}(\infty) = k_{13}^{+-}(\infty) = k_{23}^{+-}(\infty) = 0$, implying from (4.6) that two of the three charges b_i vanish, so that the solution reduces to the one-wormhole solution. This argument, which generalizes to the case of n centers, shows that any multi-center reduced three-metric constructed from the superposition (3.23) is necessarily asymptotically conical in some sheets of its maximal analytic extension.

5 Conclusion

We have generalized the construction of stationary axisymmetric multi-wormhole solutions given in [7] to gravitating field theories admitting toroidal reductions to three-dimensional gravitating sigma models. We have shown that – as in the multi-black hole case – strut singularities, necessary to restore the otherwise unequal balance between attractive and repulsive forces, always appear in this construction, except for very special configurations with an odd number of centers.

We have also reviewed the analytical continuation of the multicenter solution across the cuts associated with the wormhole mouths. Contrary to naive intuition, the resulting Riemann manifold is not two-sheeted, but has 2^n sheets interconnected by $2^{n-1}n$ wormholes, where n is the number of cuts. Some interwormhole forces change sign when going from one sheet to another adjacent sheet, so that the strut configurations are different. We have shown that, as a consequence, the maximally extended multicenter solution can never be asymptotically locally flat in all the Riemann sheets.

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